Oscillation of second-order linear difference equations with deviating arguments

by

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Abstract

Sufficient conditions which guarantee the oscillation of all solutions to the difference equation

$$\Delta^2 u(k) + \sum_{j=1}^{m} p_j(k) u(\tau_j(k)) = 0 \tag{1.1}$$

are established. Here $\Delta u(k) = u(k+1) - u(k)$, $\Delta^2 = \Delta \circ \Delta$ and the coefficients $p_j (j = 1, ..., m)$ are arbitrary sequences of nonnegative real numbers. It is to be emphasized that the deviations τ_j are subject to the restriction $\liminf_{k \to +\infty} \frac{\tau_j(k)}{k} > 0$ $(j = 1, ..., m)$ only. In the case where $j = 1$ and $\tau_1(k) \equiv k$, a discrete analogue of the well known Hille's oscillation theorem is obtained.

1. Introduction

Consider the equation

$$\Delta^2 u(k) + \sum_{j=1}^{m} p_j(k) u(\tau_j(k)) = 0, \tag{1.1}$$

where $m \geq 1$ is a natural number, $p_j : \mathrm{N} \to [0, +\infty), \tau_j : \mathrm{N} \to \mathrm{N}$ $(j = 1, ..., m)$ are functions defined on the set of natural numbers $\mathrm{N} = \{1, 2, ...\}$, i.e. sequences, $\Delta u(k) = u(k+1) - u(k)$ and $\Delta^2 = \Delta \circ \Delta$.

Throughout this paper, without further mentioning, we will suppose that

$$\lim_{k \to \infty} \tau_j(k) = +\infty \quad (j = 1, ..., m), \tag{1.2}$$

$$\sup \{p_j(i) : i \geq k\} > 0 \quad \text{for} \quad k \in \mathrm{N} \quad (j = 1, ..., m). \tag{1.3}$$

For any $n \in \mathrm{N}$ we set $\mathrm{N}_n = \{n, n+1, ...\}$.

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[1] This work was done while the author was visiting the Department of Mathematics, University of Ioannina, in the framework of the NATO Science Fellowships Programme through the Greek Ministry of National Economy.

Key words: Oscillation, difference equation, deviating arguments
AMS (MOS) Subject classifications: 39A10, 39A12

Definition 1.1. For $n\in N$ put $n_0 = \min\{\tau_j(k) : k\in N_n,\ \ j = 1, ..., m\}$. A function $u : N_{n_0} \to R$ is said to be a proper solution of (1.1) if it satisfies (1.1) on N_n and

$$\sup\{|u(i)| :\ \ i\geq k\} > 0 \quad \text{for} \quad k\in N_{n_0}.$$

Definition 1.2. We say that a proper solution $u : N\to R$ of the equation (1.1) is oscillatory if for any $n\in N$ there are $n_1, n_2 \in N_n$ such that $u(n_1)u(n_2) \leq 0$. Otherwise the proper solution is called nonoscillatory.

The present paper is concerned with the problem of oscillation of all solutions of the equation (1.1) under the assumption that the deviations $\tau_j(k)-k$ $(j = 1, ..., m)$ are not necessarily constant and may be unbounded.

The overwhelming majority of the papers devoted to oscillatory properties of difference equations treat the case where the deviations are constant. In that case (or, more generally, in the case where the deviations are bounded), a definition of the order of difference equations (see, e.g., [4, p.163]) considers the equation (1.1) as a linear difference equation of the order

$$\max\{2, \tau_j(k) - k : k\in N,\ j = 1, ..., m\} - \min\{0, \tau_j(k) - k : k\in N,\ j = 1, ..., m\}.$$

In the investigation of oscillatory properties, for the most part, it is more convenient to look at the equation (1.1) as a discrete analogue of the second order ordinary differential equation with deviating arguments

$$u''(t) + \sum_{j=1}^{m} p_j(t)u(\tau_j(t)) = 0.$$

In the case where the deviations $\tau_j(k) - k$ are unbounded, only the second approach seems natural since in that case, according to the above mentioned definition, the equation (1.1) should be considered as an infinite order difference equation. For this reason we call the equation under consideration a second order linear difference equation with deviating arguments. Of the papers treating oscillatory properties of linear difference equations in the case of unbounded deviations, we cite [5, 10, 12].

Oscillatory properties of difference equations analogous to first order ordinary differential equations with constant deviations are set forth in Chapter 7 of the monograph [4] and the references cited therein. Of the works studying oscillatory properties of linear second order difference equations we mention [1, 3, 7, 8, 11] as being most relevant to the matter of the present paper.

In section 2 some auxiliary statements are proved. In section 3 criteria for oscillation of all solutions of (1.1) are established. They imply, as a corollary, a discrete analogue of Hille's oscillation theorem [6]. The latter result also generalizes Theorem 3.4 from [1].

Everywhere below, we assume that the following conditions are fulfilled

$$\sum_{k=1}^{\infty} k \sum_{j=1}^{m} p_j(k) \ = \ +\infty \tag{1.4$_1$}$$

and

$$(1.4_2) \qquad \sum_{k=1}^{\infty} \sum_{j=1}^{m} \tau_j(k) p_j(k) = +\infty .$$

Each of these conditions is necessary for oscillation of (1.1). Indeed, if one of these sums is finite, then the equation (1.1) has a proper nonosscilatory solution. To prove this, consider the space S of all sequences $u : \mathrm{N} \to \mathrm{R}$ with the topology of pointwise convergence. Take $c_1, c_2 \in \mathrm{R}$, $c_1 < c_2$, and for any $r \in \{0,1\}$ introduce the set $\mathrm{U}_r \subset S$ by $u \in \mathrm{U}_r \Longleftrightarrow c_1 k^r \le u(k) \le c_2 k^r$ for $k \in \mathrm{N}$. If (1.4_1) is violated, then define the operator $T_0 : \mathrm{U}_0 \to S$ by

$$T_0 u(k) = \begin{cases} c_0 - \sum_{i=k-1}^{\infty} (i-k+1) \sum_{j=1}^{m} p_j(i) u(\tau_j(i)) & \text{for } k \in \mathrm{N}_{k_1} \\ T_0 u(k_1) & \text{for } k \in \mathrm{N} \setminus \mathrm{N}_1, \end{cases}$$

and if (1.4_2) is violated, then let $T_1 : \mathrm{U}_1 \to S$ be as follows:

$$T_1 u(k) = \begin{cases} c_0 k + \sum_{\ell=k_1}^{k} \sum_{i=\ell-1}^{\infty} \sum_{j=1}^{m} p_j(i) u(\tau_j(i)) & \text{for } k \in \mathrm{N}_{k_1} \\ c_0 k & \text{for } k \in \mathrm{N} \setminus \mathrm{N}_{k_1}, \end{cases}$$

where $c_0 \in (c_1, c_2)$ and $k_1 \in \mathrm{N}$. It can be easily checked that if k_1 is taken large enough, both T_0 and T_1 map U_0 and U_1, respectively, into themselves and satisfy all the conditions of the Schauder-Tychonoff theorem. The fixed point will be a nonoscillatory solution of (1.1).

Everywhere below it will be assumed that

$$(1.5) \qquad \liminf_{k \to +\infty} \frac{\tau_j(k)}{k} > 0 \quad (j = 1, \dots, m).$$

It is clear that if (1.5) is fulfilled, then (1.4_1) implies (1.4_2).

2. Auxiliary Statements

First of all, for convenience of the reader we will formulate as separate lemmas two versions of the well-known formula of summation by parts (the Abel transform) which we will use in the sequel.

Lemma 2.1. *Let $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ be two finite sequences of real numbers and $A_i = \sum_{j=1}^{i} a_j$. Then*

$$\sum_{i=1}^{n} a_i b_i = A_n b_n + \sum_{i=1}^{n-1} A_i (b_i - b_{i+1}).$$

Lemma 2.2. *Let $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$ be two infinite sequences, let the series $\sum_{i=1}^{\infty} b_i$ be convergent and $a_i B_{i+1} \to 0$ as $i \to \infty$, where $B_i = \sum_{j=i}^{\infty} b_j$. Then the* *convergence of either of the series* $\sum_{i=1}^{\infty} a_i b_i$ *and* $\sum_{i=2}^{\infty} (a_i - a_{i-1}) B_i$ *implies the convergence of the other and*

$$\sum_{i=1}^{\infty} a_i b_i = a_1 B_1 + \sum_{i=2}^{\infty} (a_i - a_{i-1}) B_i.$$

Lemma 2.3. *Let* $u : \mathrm{N}_{n_0} \to \mathrm{R}$ *be a nonoscillatory proper solution of* (1.1). *Then there exists* $k_0 \in \mathrm{N}_{n_0}$ *such that*

$$u(k)\Delta u(k) > 0 \quad \text{for} \quad k \in \mathrm{N}_{k_0}. \tag{2.1}$$

Proof. Without loss of generality we may assume that $u(k) > 0$ for $k \in \mathrm{N}_{k_1}$ with $k_1 \in \mathrm{N}_{n_0}$ sufficiently large. By (1.2) there is $k_0 \geq k_1$ such that $u(\tau_j(k)) > 0$ for $k \in \mathrm{N}_{k_0} (j = 1, ..., m)$. Hence from (1.1) according to (1.3) we have

$$\Delta u(k) \geq \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i) u(\tau_j(i)) > 0 \quad \text{for} \quad k \in \mathrm{N}_{k_0}.$$

Therefore (2.1) holds. The proof is complete.

Lemma 2.4 *Suppose that* (1.4_1), (1.5) *holds and* $u : \mathrm{N}_{n_0} \to \mathrm{R}$ *is a nonoscillatory solution of* (1.1). *Then*

$$\lim_{k \to \infty} |u(k)| = +\infty, \quad \limsup_{k \to +\infty} \frac{|u(k)|}{k} < +\infty. \tag{2.2}$$

Proof. By Lemma 2.3, from (1.1) we get

$$|u(k)| \geq \sum_{\ell=k_0}^{k-1} \sum_{i=\ell}^{\infty} \sum_{j=1}^{m} p_j(i) |u(\tau_j(i))| \quad \text{for} \quad k \geq k_0, \tag{2.3}$$

where $k_0 \in \mathrm{N}$ is sufficiently large. Using Lemma 2.1, from (2.3) we obtain

$$|u(k)| \geq (k - k_0) \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i) |u(\tau_j(i))| +$$

$$+ \sum_{\ell=k_0}^{k-1} (\ell - k_0 + 1) \sum_{j=1}^{m} p_j(\ell) |u(\tau_j(\ell))|.$$

Since $u(\tau_j(\ell)) \geq c > 0$ for $\ell \in \mathrm{N}_{k_0} (j = 1, ..., m)$, by (1.4_1) we get that the second summand tends to $+\infty$ as $k \to \infty$ which gives the first condition of (2.2).

To prove the second one, calculate

$$\Delta \left(\frac{u(k)}{k} \right) = \frac{u(k+1)}{k+1} - \frac{u(k)}{k} = \frac{k(u(k+1) - u(k)) - u(k)}{k(k+1)} =$$

$$= \frac{k \Delta u(k) - u(k)}{k(k+1)} \quad \text{for} \quad k \in \mathrm{N}_{n_0}. \tag{2.4}$$

On the other hand,

$$\Delta(k \Delta u(k) - u(k)) = (k+1) \Delta u(k+1) - (k+1) \Delta u(k) =$$

$$= (k+1)\Delta^2 u(k) \le 0 \quad \text{for} \quad k \in \mathrm{N}_{n_0}.$$

Therefore by (2.4) either (*) $\Delta\left(\frac{u(k)}{k}\right) \ge 0$ for $k \in \mathrm{N}_{n_0}$ or (**) there exists $k_0 \ge n_0$ such that $\Delta\left(\frac{u(k)}{k}\right) < 0$ for $k \in \mathrm{N}_{k_0}$.

Suppose that (*) holds. Then there is $c \in (0, +\infty)$ such that $u(k) \ge ck$ for $k \in \mathrm{N}_{n_0}$. Then from (1.1) we get

$$\Delta u(k_0) \ge c \sum_{k=k_0}^{\infty} \sum_{j=1}^{m} p_j(k) \tau_j(k)$$

which contradicts (1.4_2). The obtained contradiction shows that (**) holds. But this means that the second relation of (2.2) is true. The proof is complete.

Lemma 2.5 *Let $\varphi, \psi : \mathrm{N} \to (0, +\infty)$, ψ be nonincreasing and*

$$\lim_{k \to +\infty} \varphi(k) = +\infty \ , \tag{2.5}$$

$$\liminf_{k \to +\infty} \psi(k) \widetilde{\varphi}(k) = 0, \tag{2.6}$$

where $\widetilde{\varphi}(k) = \inf \{\varphi(s) : s \ge k, \ s \in \mathrm{N}\}$. Then there exists an increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that

$$\widetilde{\varphi}(k_i) = \varphi(k_i), \ \psi(k) \widetilde{\varphi}(k) \ge \psi(k_i) \widetilde{\varphi}(k_i) \qquad (k = 1, 2, ..., k_i, \quad i = 1, 2, ...). \tag{2.7}$$

Proof. Introduce the sets $E_i (i = 1, 2)$ in the following way:

$$k \in E_1 \Longleftrightarrow \widetilde{\varphi}(k) = \varphi(k) \ ,$$

$$k \in E_2 \Longleftrightarrow \widetilde{\varphi}(s) \psi(s) \ge \widetilde{\varphi}(k) \psi(k) \quad \text{for} \quad s \in \{1, ..., k\} \, .$$

According to (2.5) and (2.6), it is obvious that $\sup \ E_i = +\infty (i = 1, 2)$. Show that

$$\sup \ E_1 \cap E_2 = +\infty. \tag{2.8}$$

Let $k_0 \in E_2$ be such that $k_0 \notin E_1$. By (2.5) there is $k_1 > k_0$ such that $\widetilde{\varphi}(k) = \widetilde{\varphi}(k_1)$ for $k = k_0, k_0 + 1, ..., k_1$ and $\widetilde{\varphi}(k_1) = \varphi(k_1)$. Since ψ is nonincreasing, we have

$$\widetilde{\varphi}(k) \psi(k) \ge \widetilde{\varphi}(k_1) \psi(k_1) \quad \text{for} \quad k = 1, ..., k_1.$$

Therefore $k_1 \in E_1 \cap E_2$. This argument shows that (2.8) holds. But this means that the lemma is true.

Remark 2.1 The analogue of this lemma for continuous φ and ψ first was proved by R. Koplatadze in [9].

3. Main Results

Our main results will be obtained under the assumption

$$\liminf_{k \to +\infty} \frac{\tau_j(k)}{k} \ge \alpha_j \in (0, +\infty) \ (j = 1, ..., m) \ . \tag{3.1}$$

Theorem 3.1 *Let* (3.1) *be fulfilled and there exist* $\theta > 1$ *such that*

$$\liminf_{k\to+\infty} k^{1-\lambda} \sum_{i=k}^{\infty} \Big(\sum_{j=1}^{m} p_j(i)(\tau_j(i))^{\lambda} \Big) > \theta\lambda \tag{3.2}$$

for any $\lambda\in[0,1)$. *Then every proper solution of* (1.1) *is oscillatory.*

Proof. First note that the condition (3.2) (for $\lambda = 0$) implies the condition (1.4_1).

Suppose on the contrary that there exists a nonoscillatory solution $u : \mathrm{N}_{n_0}\to\mathrm{R}$ of (1.1). According to Lemmas 2.3 and 2.4 there is $k_0\in\mathrm{N}$ such that $u(k)\Delta u(k) > 0$ for $k\in\mathrm{N}_{k_0}$ and

$$\lim_{k\to\infty} |u(k)| = +\infty, \quad \limsup_{k\to+\infty} \frac{|u(k)|}{k} < +\infty\,. \tag{3.3}$$

Moreover, (2.3) is fulfilled for $k\in\mathrm{N}_{k_0}$.

Without loss of generality assume that $\tau_j(k) \geq \tau^*(k)$ for $k \in N_{k_0}$, where

$$\tau^*(k) = [\alpha k], \quad \alpha = \min\Big\{\frac{\alpha_j}{2}, 1 : j = 1,\ldots,m\Big\} \tag{3.4}$$

(here $[\alpha k]$ denotes the integral part of the number αk). Denote by Λ the set of those $\lambda\in[0,+\infty)$ for which

$$\lim_{k\to\infty} \frac{|u(k)|}{k^{\lambda}} = +\infty.$$

Using (2.3), we can ascertain that for any $\lambda\in\Lambda$ the series in the left-hand side of (3.2) is convergent (see (3.9) below). By (3.3) $\lambda_0 = \sup\Lambda$ is finite and $\lambda_0\in[0,1)$. According to (3.2) and the definition of λ_0 there exist $\varepsilon\in(0,1)$, $k_1\geq k_0$ and $\lambda^*\in[0,\lambda_0]\cap[0,1)$ such that

$$\lim_{k\to\infty} \frac{|u(k)|}{k^{\lambda^*}} = +\infty, \ \liminf_{k\to+\infty} \frac{|u(k)|}{k^{\lambda^*+\varepsilon}} = 0, \tag{3.5}$$

$$k^{1-\lambda^*} \sum_{i=k}^{\infty} \Big(\sum_{j=1}^{m} p_j(i)(\tau_j(i))^{\lambda^*} \Big) > (\lambda^* + \varepsilon)\Big(\frac{2}{\alpha}\Big)^{\varepsilon} \quad \text{for } k\in\mathrm{N}_{k_1}. \tag{3.6}$$

Indeed, suppose first that $\lambda_0 > 0$. If we choose ε so that

$$0 < \frac{\varepsilon\left(\frac{2}{\alpha}\right)^{\varepsilon}}{\theta - \left(\frac{2}{\alpha}\right)^{\varepsilon}} \leq \frac{\lambda_0}{2},$$

then we will have $\mathrm{P}(\Lambda) > (\lambda+\varepsilon)\left(\frac{2}{\alpha}\right)^{\varepsilon}$ for $\lambda\in\left[\frac{\lambda_0}{2},\lambda_0\right)$, where $\mathrm{P}(\lambda)$ is the left-hand side of (3.2). Then we can choose $\lambda^*\in\left[\frac{\lambda_0}{2},\lambda_0\right)$ and $k_1\geq k_0$ such that (3.5) and (3.6) be fulfilled. In the case $\lambda_0 = 0$ the situation is even simpler since we have the unique choice $\lambda^* = \lambda_0 = 0$.

According to (3.1) and (3.4), k_1 may be supposed large enough for the inequality

$$\frac{\tau^*(k)}{k} \geq \frac{2}{3}\alpha \quad \text{for } k\in\mathrm{N}_{k_1}. \tag{3.7}$$

In view of (3.5), all conditions of Lemma 2.5 are fulfilled with $\varphi(k)\equiv (\tau^*(k))^{-\lambda^*}\ |u(\tau^*(k))|$ and $\psi(k)\equiv(\tau^*(k))^{-\varepsilon}$. Therefore there exists an increasing sequence of natural numbers $\{k_i\}_{i=2}^{+\infty}$ such that

$$\tau^*(k_2) > k_1, \quad (\tau^*(k_i))^{-\varepsilon}\widetilde{\varphi}(k_i)\le(\tau^*(k))^{-\varepsilon}\widetilde{\varphi}(k) \text{ for } k_1\le k\le k_i, \tag{3.8}$$

$$\widetilde{\varphi}(k_i) = (\tau^*(k_i))^{-\lambda^*}\ |u(\tau^*(k_i))| \text{ for } i\ge 2, \tag{3.9}$$

where $\widetilde{\varphi}$ is defined in the lemma 2.5.

On the other hand, since $\alpha \le 1$ we have $\tau^*(N_k) = N_{\tau^*(k)}$. Moreover, $\tau_j(k) \ge \tau^*(k)$ for large k $(j = 1, \dots, m)$. Therefore

$$\inf_{i\ge k} \frac{|u(\tau_j(i))|}{(\tau_j(i))^{\lambda^*}} \ge \inf_{i\ge k} \frac{|u(\tau^*(i))|}{(\tau^*(i))^{\lambda^*}} = \widetilde{\varphi}(k) \quad (j = 1, \dots, m)$$

for large k.

Hence using (2.3), we have

$$|u(\tau^*(k_i))| \ge \sum_{k=k_1}^{\tau^*(k_i)-1} \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i)\ |u(\tau_j(i))| \ge$$

$$\ge \sum_{k=k_1}^{\tau^*(k_i)-1} \sum_{i=k}^{+\infty} \sum_{j=1}^{m} p_j(i)\ (\tau_j(i))^{\lambda^*} \inf_{i\ge k} \frac{|u(\tau_j(i))|}{(\tau_j(i))^{\lambda^*}} \ge$$

$$\ge \sum_{k=k_1}^{\tau^*(k_i)-1} \widetilde{\varphi}(k) \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i)(\tau_j(i))^{\lambda^*} \text{ for } i\ge 2. \tag{3.10}$$

Hence, taking into consideration (3.6)–(3.8), we obtain

$$|u(\tau^*(k_i)|\ge(\lambda^* + \varepsilon)\left(\frac{2}{\alpha}\right)^{\varepsilon} \sum_{k=k_1}^{\tau^*(k_i)-1} \widetilde{\varphi}(k)k^{\lambda^*-1} =$$

$$= (\lambda^* + \varepsilon)\left(\frac{2}{\alpha}\right)^{\varepsilon} \sum_{k=k_1}^{\tau^*(k_i)-1} \widetilde{\varphi}(k)(\tau^*(k))^{-\varepsilon}k^{\lambda^*-1+\varepsilon}\left(\frac{\tau^*(k)}{k}\right)^{\varepsilon} \ge$$

$$\ge \left(\frac{4}{3}\right)^{\varepsilon}(\lambda^* + \varepsilon)\widetilde{\varphi}(k_i)(\tau^*(k_i))^{-\varepsilon} \sum_{k=k_1}^{\tau^*(k_i)-1} k^{\lambda^*-1+\varepsilon} \text{ for } i\ge 2\,. \tag{3.11}$$

We were able to write the latter inequality in view of the fact that $\tau^*(k)\le k$ for $k\in\mathrm{N}$.

Since for any $\lambda\ge 0$ and $\ell, n\in\mathrm{N}$ $(\ell\le n)$

$$\sum_{k=\ell}^{n} k^{\lambda} \ge \sum_{k=\ell}^{n} \int_{k-1}^{k} x^{\lambda}dx = \frac{1}{\lambda+1}\sum_{k=\ell}^{n}(k^{\lambda+1} - (k-1)^{\lambda+1}) =$$

$$= \frac{1}{\lambda+1}\left(n^{\lambda+1} - (l-1)^{\lambda+1}\right)\,,$$

we have

$$\sum_{k=k_1}^{\tau^*(k_i)-1} k^{\lambda^*-1+\varepsilon} \geq \frac{1}{\lambda^*+\varepsilon}\ \left((\tau^*(k_i)-1)^{\lambda^*+\varepsilon} - k_1^{\lambda^*+\varepsilon}\right).$$

Therefore, taking into consideration (3.9) and the definition of $\widetilde{\varphi}$, from (3.11) we obtain

$$u(\tau^*(k_i)) \geq u(\tau^*(k_i)) \left(\frac{4}{3}\right)^{\varepsilon} \left(\frac{\tau^*(k_i)-1}{\tau^*(k_i)}\right)^{\lambda^*+\varepsilon} \left(1-(\tau^*(k_i)-1)^{-\lambda-\varepsilon}\, k_1^{\lambda^*+\varepsilon}\right)$$

for $i \geq 2$.

Since $\left(\frac{4}{3}\right)^{\varepsilon} > 1$ and $\tau^*(k_i) \to +\infty$ as $i \to \infty$, the last inequality is impossible. This is a contradiction and the proof is complete.

Theorem 3.2 *Let the conditions* (1.4_1) *and* (3.1) *be fulfilled and there exist* $\theta > 1$ *such that*

$$\liminf_{k \to +\infty} k \sum_{i=k}^{\infty} \left(\sum_{j=1}^{m} p_j(i) \left(\frac{\tau_j(i)}{i}\right)^{\lambda} \right) > \theta\lambda(1-\lambda) \tag{3.12}$$

for any $\lambda \in [0,1)$. *Then all proper solutions of* (1.1) *are oscillatory.*

Proof. It suffices to show that (3.2) holds for any $\lambda \in [0,1)$. If the series in the left-hand side of (3.2) is divergent, then (3.2) is obviously true. Therefore we will assume that the series is convergent. The convergence of the latter series implies the convergence of the series in (3.12) and

$$\lim_{k \to \infty} k^{\lambda} \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i) \left(\frac{\tau_j(i)}{i}\right)^{\lambda} = 0\,.$$

So using Lemma 2.2, from (3.12) we get

$$\begin{aligned}
k^{1-\lambda} \sum_{i=k}^{\infty} \sum_{j=1}^{m} p_j(i)(\tau_j(i))^{\lambda} &= k^{1-\lambda} \sum_{i=k}^{\infty} i^{\lambda} \left(\sum_{j=1}^{m} p_j(i) \left(\frac{\tau_j(i)}{i}\right)^{\lambda} \right) = \\
&= k^{1-\lambda} k^{\lambda} \sum_{i=k}^{\infty} \left(\sum_{j=1}^{m} p_j(i) \left(\frac{\tau_j(i)}{i}\right)^{\lambda} \right) + \\
&+ k^{1-\lambda} \sum_{i=k+1}^{\infty} \left(i^{\lambda} - (i-1)^{\lambda}\right) \sum_{s=i}^{\infty} \left(\sum_{j=1}^{m} p_j(s) \left(\frac{\tau_j(s)}{s}\right)^{\lambda} \right) > \\
&> \theta\lambda(1-\lambda) + \theta\lambda(1-\lambda) k^{1-\lambda} \sum_{i=k+1}^{\infty} \left(i^{\lambda} - (i-1)^{\lambda}\right) i^{-1} = \\
&= \theta\lambda(1-\lambda) \left(1 + k^{1-\lambda} \sum_{i=k+1}^{\infty} i^{-1} \left(i^{\lambda} - (i-1)^{\lambda}\right) \right)
\end{aligned}$$

for $k \in \mathrm{N}_{k_0}$, $\lambda \in [0,1)$, where $k_0 \in \mathrm{N}$ is sufficiently large.

Since $\quad i^{-1}\left(i^{\lambda}-(i-1)^{\lambda}\right)=\lambda i^{-1}\int_{i-1}^{i} x^{\lambda-1}dx \geq$

$$\geq \lambda i^{\lambda-2} \text{ for } i\in\mathrm{N},\ \lambda\in[0,1),$$

it suffices to show that

$$\liminf_{k\to+\infty} k^{1-\lambda}\sum_{i=k+1}^{\infty} i^{\lambda-2} \geq \frac{1}{1-\lambda}\,. \tag{3.13}$$

We have

$$\sum_{i=k+1}^{\infty} i^{\lambda-2} \geq \sum_{i=k+1}^{\infty}\int_{i}^{i+1} x^{\lambda-2}dx = \int_{k+1}^{+\infty} x^{\lambda-2}dx = \frac{(k+1)^{\lambda-1}}{1-\lambda}\,.$$

Taking the latter inequality into account, we get convinced that (3.13) is true. Therefore there exists $\theta_1\in(0,\theta)$ such that

$$\liminf_{k\to+\infty} k^{1-\lambda}\sum_{i=k}^{+\infty}\sum_{j=1}^{m} p_j(i)(\tau_j(i))^{\lambda} > \theta_1\lambda(1-\lambda)\left(1+\frac{\lambda}{1-\lambda}\right)=\theta_1\lambda$$

i.e., (3.2) is fulfilled. The proof if complete.

Theorem 3.2 immediately implies the following

Corollary 3.1 *Let* (3.1) *be fulfilled and*

$$p_j(k)\geq c_j p(k) \quad \textit{for} \quad k\in\mathrm{N} \quad (j=1,\ldots,m),$$

where $p:\mathrm{N}\to[0,+\infty)$ *and* $c_j\in(0,+\infty)$ $(j=1,\ldots,m)$. *Let, moreover,*

$$\liminf_{k\to+\infty} k\sum_{i=k}^{\infty} p(i) > \max\left\{\Big(\sum_{j=1}^{m} c_j\ \alpha_j^{\lambda}\Big)^{-1}\lambda(1-\lambda):\ \lambda\in[0,1]\right\}.$$

Then all proper solutions of (1.1) *are oscillatory.*

For the equation

$$\Delta^2 u(k)+p(k)u(\tau(k))=0 \tag{3.14}$$

the corollary reads as

Corollary 3.2. *Let* $p:\mathrm{N}\to[0,+\infty),\tau:\mathrm{N}\to\mathrm{N}$,

$$\liminf_{k\to+\infty}\frac{\tau(k)}{k}=\alpha\in(0,+\infty),$$

and

$$\liminf_{k\to+\infty} k\sum_{i=k}^{\infty} p(i) > \max\left\{\alpha^{-\lambda}\lambda(1-\lambda):\lambda\in[0,1]\right\}\,. \tag{3.15}$$

Then all proper solutions of (3.14) *are oscillatory.*

In the case where $\alpha=1$, we get the following discrete analogue of Hille's well-known oscillation theorem for ordinary differential equations [6].

Corollary 3.3 *Let k_0 be an integer, $p:\mathrm{N}\to[0,+\infty)$. Then the condition*

$$\liminf_{k\to+\infty} k\sum_{i=k}^{\infty} p(i) > \frac{1}{4} \tag{3.16}$$

is sufficient for all proper solutions of the equation

$$\Delta^2 u(k) + p(k)u(k-k_0) = 0, \quad k > k_0,$$

to be oscillatory.

This corollary for $k_0 = -1$ is proved in [1] (Theorem 3.4).

Remark 3.1 As in the case of ordinary differential equations, the constant $\frac{1}{4}$ in (3.16) is optimal and the strict inequality cannot be replaced by the nonstrict one. More than that, the same is true for the condition (3.15) as well. To ascertain this, denote by c the right-hand side of (3.15), and by λ_0 the point where this maximum is achieved. The sequence $u(k) = k^{\lambda_0}$ obviously is a nonoscillatory proper solution of the equation

$$\Delta^2 u(k) + p(k)u([\alpha k]) = 0,$$

where $p(k) = -\frac{\Delta^2(k^{\lambda_0})}{[\alpha k]^{\lambda_0}}$ and $[a]$ denotes the integer part of a. It can be easily calculated that

$$p(k) = \frac{c}{k^2} + o\left(\frac{1}{k^2}\right) \quad \text{as} \quad k\to\infty\ .$$

Hence for arbitrary $\varepsilon > 0$ $p(k) \geq \frac{c-\varepsilon}{k^2}$ for $k\in\mathrm{N}_{k_0}$ with $k_0\in\mathrm{N}$ sufficiently large. Using the inequality $\sum_{i=k}^{\infty} i^{-2} \geq k^{-1}$ and the arbitrariness of ε, we obtain

$$\liminf_{k\to+\infty} k\sum_{i=k}^{\infty} p(i) \geq c\ .$$

This limit can not be greater than c by Corollary 3.2. Therefore it equals c and (3.15) is violated.

References

1. S. S. Cheng, T. Ch. Yan and H. J. Li, Oscillation criteria for second order difference equation. *Funkcialaj Ekvacioj*, **34** (1991), 223-239.
2. R. E. Edwards, Functional analysis. Theory and applications. Holt, Rinehart and Winston, Chicago - San Fransisco - Toronto, 1965.
3. S. R. Grace and B. S. Lalli, Oscillation theorems for second order delay and neutral difference equations. *Utilitas Mathematica*, **45** (1994), 197-211.
4. I. Györi and G. Ladas, Oscillation theory of delay differential equations with applications. *Oxford University Press*, Oxford, 1991.
5. D. Hankerson and G. A. Harris, Oscillation of first and second order delay difference equations. *Dynamic Systems and Applications*, **4** (1995), 251-262.
6. E. Hille, Nonoscillation Theorems. *Trans. Amer. Math. Soc.* **64** (1948), 234-252.
7. D. B. Hinton and R. T. Lewis, Spectral analysis of second order difference equations. *J. Math. Anal. Appl.*, **63** (1978), 421-438.
8. J. W. Hooker and W. T. Patula, Riccati type trasformations for second order linear difference equations. *J. Math. Anal. Appl.*, **82** (1981), 451-462.

9. R. Koplatadze, On oscillatory properties of solutions of functional differential equations. *Mem. Differential Equations Math. Phys.*, **3** (1994), 1-177.
10. I.-G. E. Kordonis and Ch. G. Philos, Oscillations and nonoscillation in linear delay or advanced difference equations. *Math. Comput. Modelling* **27** (1998), No 7, 11-21.
11. M. K. Kwong, J. W. Hooker and W. T. Patula, Riccati type transformations for second order linear difference equations, II. *J. Math. Anal. Appl.*, **107** (1985), 182-196.
12. B. G. Zhang and Ch. J. Tian, Oscillation criteria for difference equations with unbounded delay. *Computers Math. Applic.* **35** (1998), No 4, 19-26.

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